The Dynamic Properties of Inflation Targeting Under Uncertainty*

Maria Demertzis†
De Nederlandsche Bank and University of Amsterdam

Nicola Viegi‡
University of Cape Town and ERSA

April 2008

Abstract

We study the implications of uncertainty for inflation targeting in a dynamic set-up. Using Svensson's inflation forecast targeting model, we compare the Brainard conservative principle to a more active monetary policy rule, derived from a two-step optimisation procedure. Our analysis points to a trade-off between the ability to control expectations and the introduction of greater variability in the system. We show that Brainard´s attenuation principle is optimal only in a backward looking set-up where there is no role for expectations in the determination of inflation equilibrium. On the other hand, we show that in a forward looking model, Brainard's conservative principle may produce instability, because of its inability to control expectations. A more aggressive rule, of the type we put forward in this paper, can instead provide greater stability because it provides for a better and more direct management of expectations, despite the uncertainty in the transmission parameters. In that respect, we show that there are conditions under which the benefits of tying down expectations, more than compensate the costs of having to overuse the instrument.

J.E.L. Classification: E42, E52

*(older version DNB Staff Report No. 113) Views expressed are our own and do not necessarily reflect those of the DNB. We would like to thank Peter van Els, Bob Chirinko and seminar participants at the University of Glasgow, the Hungarian Academy of Science and the EEA03 in Stockholm for comments and suggestions. The usual disclaimer naturally applies.

†Corresponding author: m.demertzis@dnb.nl, Economics & Research, De Nederlandsche Bank, P.O. Box 98, 1000 AB, Amsterdam, The Netherlands, tel: +31 20 524 524 2506, www1.fee.uva.nl/toe/content/people/demertzis.shtm

‡nicola.viegi@uct.ac.za, School of Economics, University of Cape Town, Private Bag, Rondebosch, 7700, Cape Town, South Africa, tel: +27 (21) 650 2763, fax: +27 (21) 650 2854.

Keywords: Inflation Targeting, Parameter Uncertainty, Two-Step Target, Dynamic Models

"....insofar as it is possible for the Central Bank to affect expectations this should be an important tool of stabilization policy...." Woodford (2003).

1 Introduction

Our motivation stems from the observation that in the presence of parameter uncertainty in backward looking models (Brainard, 1967), inflation expectations formed are not in line with the target the Central Bank (CB) sets, but fall short of it in proportion to the degree of the existing uncertainty. As inflation expectations play no role in such models, this does not affect the 'optimality' of the policy. In the presence of parameter uncertainty, it is therefore optimal for policy makers to apply cautious rules, very much in the spirit of Brainard's recommendation, but then live with the fact that their inflation objective will be seldom achieved. Choosing to apply any other rule that might perhaps bring expectations closer to the target introduces only instability in the system and is de facto welfare reducing.

The role of expectations in determining final outcomes is very different however, when considering forward looking models. So, the natural question that follows is whether, in the presence of parameter uncertainty, failing to align expectations to the target in such systems is equally innocuous to the final outcome. Solving forward looking models analytically in the presence of parameter uncertainty is not well established in the literature, so in order to analyze the issue we will move away from an optimization framework and examine instead how the stability of the system is affected, when applying different degrees of policy 'aggression'. We will compare a rule that factors in uncertainty and therefore justifies attenuated policies, to a rule that aims explicitly at aligning expectations to the target but introduces greater variability in the system. We will show that in backward looking models, where there is no role for expectations, the stability of the system benefits from policies that apply caution, as already mentioned. In forward looking models however, where expectations by definition play an active role, we will show that attenuated policies introduce instability in the system. By contrast, a rule that aims explicitly at aligning expectations to the target is quicker to provide stable outcomes.

Previous attempts to analyze the issue in a dynamic setting have confined themselves to a backward-looking framework for two main reasons: forward-looking models do not appear to replicate real data patterns accurately (Rudd and Whelan, 2006) and more importantly, it is not clear how to optimize forward-looking models with multiplicative uncertainty. On the first point, the issue is not one of realism but of plausibility. Modern economic policy is developed around the idea that the private sector "reacts" to economic policy actions, and this reaction must be taken into consideration when determining the optimal policy rule. This is particularly evident when looking at the inflation targeting debate. Proponents of inflation targeting have emphasized that the most important aspect of such a regime is that it aims precisely at bringing expectations closer to the CB's inflation objective (Bernanke and Mishkin, 1997, Bernanke et al, 1999, Demertzis and Viegi, 2008). A number of studies also support this statement empirically (see Johnson, 2002, Mishkin and Schmidt-Hebbel, 2001, 2007, Levin *et al*, 2004, and Gürkaynak et al, 2006). In fact, the argument goes further than that and explains that any *ex ante* target announced that is not hit also *ex post* reduces the credibility of monetary policy and therefore the efficiency of the regime itself (Demertzis and Viegi, 2007). The only way to make the target relevant is to adjust the rule such that the inflation target is hit the majority of times. The rule that we will apply in our example is designed to explicitly pin down expectations to the objective set by the Central Bank, by optimally 'ignoring' the level of uncertainty. And it is important that this is done in an efficient framework so that it is possible to communicate to the public the rationale behind it. This rule proves indeed beneficial, in the sense of more stable, in a system where actual policy outcomes are subject to expectations feedback.

The paper is organized as follows: Section 2 will describe a standard backward-looking model and section 3 applies a two-step algorithm with aims directly at tieing expectations to the target. Section 4 will demonstrate numerically, why an algorithm that ties down expectations is irrelevant in a backward-looking model. By contrast, Section 5 will demonstrate (but not prove) that the ability to tie down expectations generates greater stability when the economy is forward-looking.

2 The Model

2.1 Certainty in the Parameters

We assume a dynamic IS-AS model (Svensson 1999a, b and Söderström, 2002). The model is applied extensively in the inflation forecast targeting literature and has the characteristic that monetary policy affects the level of inflation with a two-period time lag. It is represented by the following two equations:

$$\pi_{t+1} = \pi_t + a\,(y_t) + \varepsilon_{t+1} \tag{1}$$

$$y_{t+1} = b\,(y_t) - c\left(i_t - \pi^e_{t+1|t}\right) + \eta_{t+1} \tag{2}$$

where y_t is the log of aggregate output in deviation from potential output ($Y_t - Y^*$), π_t is the inflation rate, $i_t - \pi^e_{t+1|t}$ is the policy maker's intended deviation of the real interest from its neutral level, and ε_t and η_t are white noise random shocks. Furthermore, the coefficients satisfy, $a > 0$, $0 < b < 1$, $c > 0$. We consider a standard sequential game between the Central Bank and the private sector. The latter forms expectations first, which constitute the basis on which to base wage negotiations, (i.e. $w = \pi^e$). A shock occurs next and the CB reacts by choosing that interest rate which optimizes the conditional expectation of its loss function, expressed in terms of deviations of inflation and output from their respective targets.

The main characteristic of this model is that monetary policy can only operate on the basis of the conditional inflation forecast of two periods ahead, which is given by:

$$\pi_{t+2} = \pi_{t+1} + a\left(y_{t+1}\right) + \varepsilon_{t+2} \tag{3}$$

Substituting (1) and (2) in (3) we have:

$$\pi_{t+2} = \pi_t + a\left(y_t\right) + \varepsilon_{t+1} + a\left[b\left(y_t\right) - c\left(i_t - \pi_{t+1|t}\right) + \eta_{t+1}\right] + \varepsilon_{t+2} \tag{4}$$

We observe that expected inflation at time $t+1$ is predetermined and following (1) equal to:

$$E_t\pi_{t+1} = \pi_t + a\left(y_t\right) \tag{5}$$

We can therefore, re-write (4), as

$$\pi_{t+2} = \pi_t + a\left(1+b\right)y_t - ac\left(i_t - \pi_t - ay_t\right) + a\eta_{t+1} + \varepsilon_{t+1} + \varepsilon_{t+2} \tag{6}$$

or,

$$\pi_{t+2} = \left(1+ac\right)\pi_t + a\left(1+b+ac\right)y_t - aci_t + a\eta_{t+1} + \varepsilon_{t+1} + \varepsilon_{t+2} \tag{7}$$

The Central Bank aims to stabilize inflation and the output gap. In other words, it aims to minimize the following intertemporal loss function:

$$\min_{\{i_\tau\}_{\tau=t}^{\infty}} E_t \sum_{\tau=t}^{\infty} \delta^\tau L\left(\pi_\tau, y_\tau\right) \tag{8}$$

We will assume for simplicity that the Central Bank is following a strict inflation targeting objective and given the backward looking nature of the model, minimizes the period-by-period problem. The objective function (8), thus simplifies to:

$$\min_{i_t} E\left(L\right) = \delta^2 E_t\left[\frac{1}{2}\left(\pi_{t+2} - \pi^*\right)^2\right] \tag{9}$$

The solution to this linear quadratic problem produces the following rules:

$$i_t = \pi_t + ay_t + \frac{1}{ac}\left(\pi_t - \pi^*\right) + \frac{\left(1+b\right)}{c}y_t \tag{10}$$

or

$$i_t = \pi_{t+1|t} + \frac{1}{ac}\left(\pi_t - \pi^*\right) + \frac{\left(1+b\right)}{c}y_t \tag{11}$$

and private sector expectations match the announced target, i.e.:

$$\pi^e_{t+2|t} = \pi^* \tag{12}$$

2.2 Multiplicative Uncertainty

Following Brainard's (1967) analysis, we consider next a simple form of transmission (parameter) uncertainty (see Svensson 1999a). Parameter c in equation (2) is assumed now to be drawn randomly from $c_t \rightarrow N\left(\bar{c}, \sigma_c^2\right)$. This implies that there is uncertainty in year t when the instrument is chosen, about its effect on the policy variable (transmission uncertainty). The CB's objective is now modified as follows:

$$\begin{aligned} \min_{i_t} E(L) &= \delta^2 E_t\left[\frac{1}{2}\left(\pi_{t+2}-\pi^*\right)^2\right] && (13) \\ &= \delta^2\left[\frac{1}{2}\left(\bar{\pi}_{t+2}-\pi^*\right)^2+\sigma_{\pi_{t+2}}^2\right] && (14) \end{aligned}$$

assuming zero covariances between the shocks and the uncertain parameter. Based on equations (6) and (7), we can re-write the first and second moments of the distribution of inflation at time $t+2$ as follows:

$$\begin{aligned} E(\pi_{t+2}) &= (1+a\bar{c})\,\pi_t + a\,(1+b+a\bar{c})\,y_t - a\bar{c}i_t \\ var(\pi_{t+2}) &= a^2\left(i_t-\pi_t-ay_t\right)^2\sigma_c^2+2\sigma_\varepsilon^2+a^2\sigma_\eta^2 && (15) \end{aligned}$$

This allows us to re-write (14) in terms of the instrument:

$$\min_{i_t} E(L) = \frac{\delta^2}{2}\left[\left(\bar{\pi}_{t+2}-\pi^*\right)^2+a^2\left(i_t-\pi_t-ay_t\right)^2\sigma_c^2+2\sigma_\varepsilon^2+a^2\sigma_\eta^2\right] \quad (16)$$

Solving the objective function subject to the economic set-up produces the following monetary policy reaction function and expected inflation, respectively:

$$\begin{aligned} i_{t,BR} &= \pi_{t+1|t}+\frac{\bar{c}}{a\left(\bar{c}^2+\sigma_c^2\right)}\left(\pi_t-\pi^*\right)+\frac{\bar{c}}{\left(\bar{c}^2+\sigma_c^2\right)}(1+b)y_t && (17) \\ \pi_{t+2|t,BR}^e &= \frac{\sigma_c^2}{\left(\bar{c}^2+\sigma_c^2\right)}\pi_t+\frac{\bar{c}^2}{\left(\bar{c}^2+\sigma_c^2\right)}\pi^*+\frac{\sigma_c^2}{\bar{c}^2+\sigma_c^2}a(1+b)y_t && (18) \end{aligned}$$

Solutions (17) and (18) confirm the traditional properties of Brainard's analysis. Uncertainty in the transmission mechanism increases the penalty associated with using the instrument. Therefore, an increase in uncertainty immobilizes the instrument, as well as reducing the relevance of the inflation target in determining inflation. This in turn, makes the value added of an inflation targeting regime subject to the level of uncertainty present in the economy. The higher the level of uncertainty, the more distant expected inflation will be from the desired target. In that respect, the private sector discounts the monetary authority's ability to get to its intended level of inflation, and by implication making its objectives more difficult to attain.

In view of (18), we look for a policy rule (derived in a *two-step* optimization procedure described in detail in Appendix A) that allows now for private expectations to match the authority's inflation objective, thereby reactivating the relevance of the inflation target. Therefore, the ability to tie down expectations to the pre-announced target is the big advantage of the two-step rule. Equations (19) and (20) show the rules derived following the two-step procedure:

$$i_{t,TS} = \pi_{t+1|t} + \frac{1}{a\bar{c}}\left(\pi_t - \pi^*\right) + \frac{(1+b)}{\bar{c}} y_t \quad (19)$$

$$\pi^e_{t+2|t,TS} = \pi^* \quad (20)$$

The rules achieved are similar to those attained with no uncertainty (with c replaced by $\bar{c}$). This demonstrates that by varying the target optimally, uncertainty in the transmission process is neutralized and the private sector forms expectations that are again determined by the announcement of the Central Bank. This however, comes at a cost of introducing greater variability in the system. By definition, it is the case that as (17) and (18) are the two rules that optimize (14), average welfare deteriorates by applying the two-step solutions given by (19) and (20). However, as we are not concerned with optimality here, we examine next how this rule favours, by comparison to the 'optimal' rule, in terms of stability instead. We will show that when expectations play no role in the economy then the TS rule generates instability in the system whereas Brainard caution produces stable results. By contrast, when expectations are determined endogenously in the system (i.e. in a forward looking model), then the ability to tie down expectations is more likely to lead to stability in the system. We show this next.

3 Expectations and Stability

We examine next the ability of these two rules to generate stability in the system. First, we apply them in the backward looking system defined in (1) and (2) and then to a forward looking system.

3.1 A Backward-Looking Model

The two policy rules derived above are of the Taylor-type, in which the current nominal interest rate reacts to deviations of output and inflation from some target (for output this is its long run growth path, and for inflation the target decided by the CB itself), i.e.:

$$i_t = \bar{i}_t + \phi_\pi\left(\pi_t - \pi^*\right) + \phi_y\left(y_t\right) \quad (21)$$

where $\bar{i}_t$ is an exogenous and possibly time varying intercept[1]. Parameters ϕ_π and ϕ_y take the following values for the two rules under consideration, respectively:

[1] Svensson (1999) considers that the average federal funds rate.

$$
\begin{array}{llll}
i_{t,BR}: & \phi_{\pi,BR} = \frac{\bar{c}}{a(\bar{c}^2+\sigma^2)}, & \phi_{y,BR} = \frac{\bar{c}}{\bar{c}^2+\sigma^2}(1+b) \\
i_{t,TS}: & \phi_{\pi,TS} = \frac{1}{a\bar{c}}, & \phi_{y,TS} = \frac{(1+b)}{\bar{c}}
\end{array}
$$

For any (unknown) realization of c at time t, the system can then be written as:

$$
\begin{aligned}
\pi_{t+1} &= \pi_t + a\,(y_t) + \varepsilon_{t+1} && (22)\\
y_{t+1} &= b\,(y_t) - c\left(\bar{i}_t + \phi_\pi\,(\pi_t - \pi^*) + \phi_y\,(y_t) - \pi_{t+1|t}\right) + \eta_{t+1} \quad \text{or,} \\
y_{t+1} &= b\,(y_t) - c\left(\bar{i}_t + \phi_\pi\,(\pi_t - \pi^*) + \phi_y\,(y_t) - \pi_t - a\,(y_t)\right) + \eta_{t+1} && (23)
\end{aligned}
$$

In matrix form this is represented as:

$$
\mathbf{z}_{t+1} = \mathbf{A} * \mathbf{z}_t + \mathbf{B} * \pi^* + \mathbf{u}_{t+1} \tag{24}
$$

where $\mathbf{z}_t' = \begin{bmatrix} \pi_t & y_t \end{bmatrix}$, $\mathbf{u}_t' = \begin{bmatrix} \varepsilon_t & \eta_t \end{bmatrix}$ and

$$
\mathbf{A} = \begin{bmatrix} 1 & a \\ c\,(1-\phi_\pi) & b + c\left(a - \phi_y\right) \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ c\phi_\pi \end{bmatrix} \tag{25}
$$

Appendix B discusses this in detail.

Proposition 1 *The Brainard attenuation principle serves best an environment in which there is no role for expectations, in terms of generating stability.*

Proof 1: A stable backward-looking system requires both eigenvalues of $\mathbf{A}$ to be inside the unit circle (see Holly and Hughes Hallett, 1989). For a 2×2 matrix, and given our parameter restrictions, this requires:

$$
\left| \frac{tr(\mathbf{A}) \pm \sqrt{tr(\mathbf{A})^2 - 4\det(\mathbf{A})}}{2} \right| < 1 \tag{26}
$$

Relation (26) gives a set of sufficient conditions for the stability of the system. A necessary condition for stability is:

$$
\begin{aligned}
\frac{tr(\mathbf{A}) + \sqrt{tr(\mathbf{A})^2 - 4\det(\mathbf{A})}}{2} &< 1 \\
\sqrt{tr(\mathbf{A})^2 - 4\det(\mathbf{A})} &< 2 - tr(\mathbf{A})
\end{aligned}
$$

or,

$$
tr(\mathbf{A}) - \det(\mathbf{A}) < 1 \tag{27}
$$

and

$$
\begin{aligned}
\frac{tr(\mathbf{A}) - \sqrt{tr(\mathbf{A})^2 - 4\det(\mathbf{A})}}{2} &< 1 \\
-\sqrt{tr(\mathbf{A})^2 - 4\det(\mathbf{A})} &< 2 - tr(\mathbf{A})
\end{aligned}
$$

But if $0 < \sqrt{tr(\mathbf{A})^2 - 4\det(\mathbf{A})} < 2 - tr(\mathbf{A})$ then $-\sqrt{tr(\mathbf{A})^2 - 4\det(\mathbf{A})} < 2 - tr(\mathbf{A})$ will also be true. So we only need (27) to hold which can be rewritten as:

$$1 + a\bar{c} - \frac{\bar{c}^2}{\bar{c}^2 + \sigma_c^2} < 1 \tag{28}$$

From (28) we can study the behavior of the LHS for a change in the variance parameter, which is the one that differentiates the Brainard policy rule from out two-step procedure. The LHS is clearly an increasing function of σ_c^2, for any realization of $c > 0$. Moreover, if $\frac{\bar{c}^2}{\bar{c}^2+\sigma_c^2} > a\bar{c}$, then the inequality is not satisfied, and uncertainty increases the likelihood of monetary policy producing instability. Further numerical analysis of condition (26) confirms this observation[2].

3.1.1 A graphic representation

The above results seem to confirm that it is theoretically possible under Brainard uncertainty to use a "two-step inflation target" to stabilize inflation forecast around a preferred inflation path. But this only arises at the expense of greater variability in the system. We perform a Monte Carlo study of the property of the system under the two rules.
The following table shows the results of 30,000 stochastic simulations of our model under the two regimes , à la Brainard, and *two-step* inflation targeting. A random shock ε is drawn from a $N(0,1)$ distribution, while parameter c is drawn from a $N\left(0.5, 0.5^2\right)$ distribution. The inflation target π^* is assumed to be 2. As this simulation is performed for illustrative purposes, we have chosen values for the parameters which present a clearer picture, without affecting the results: the sensitivity of inflation to output is thus $a = 0.8$ and the income inertia parameter $b = 0.4$. We present the mean, standard deviation and the maximum and minimum values of a ten period average value of the variables i, π and y derived. The first three columns show the results for the Brainard case and the last three, those under the *two-step* inflation targeting procedure.

TABLE 2 : Monte Carlo Simulations:

	i_{BR}	π_{BR}	y_{BR}	i_{TS}	π_{TS}	y_{TS}
Mean	-0.27	1.23	0.23	1.84	2.11	1.15
St. Dev.	0.31	0.37	0.01	16.79	5.48	6.83
Max	1.36	3.20	0.67	271.07	91.48	107
Min.	-6.82	-6.19	-0.36	-183	-62.59	-106

The two-step strategy produces instability as is also shown in figure 1 where we polt one indicative realization. In the context of a backward-looking model, the Brainard rule appears to be more successful in terms of both stability as

[2]Maple workbook available from the authors.

well as bringing inflation closer to the target, at least in the long run. The two-step rule, although designed to reach the target on average, introduces instability in the system through the excessive use of the instrument (note the difference in scale). The two-step rule under-performs always, regardless of the parameterization applied.

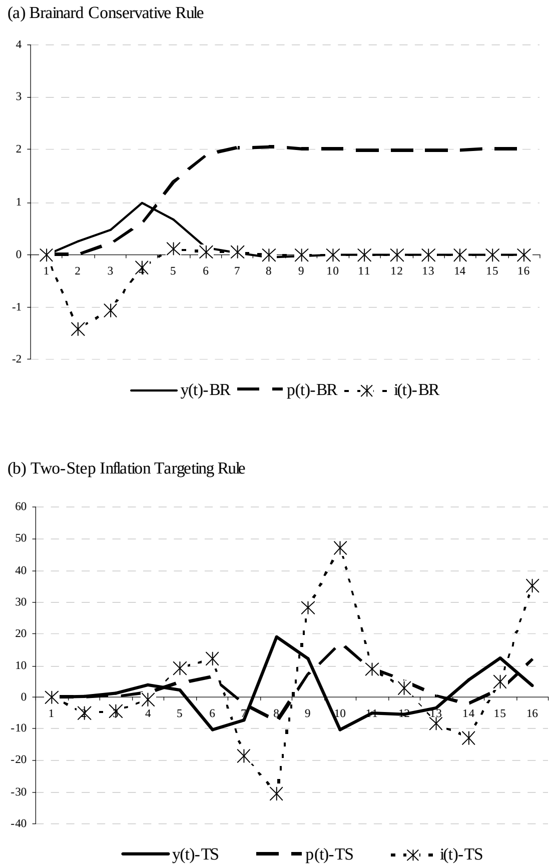

Figure 1: Stability in a backward-looking model

This raises an important issue about the context in which monetary policy rules are derived. The two-step targeting rule is derived in order to hit the inflation target with regularity. The main advantage of this strategy is that monetary policy is more transparent and therefore more credible because it is more likely to reach the announced target. In contrast to that, a monetary policy rule which emphasizes the risk over the achievement of the target (as in Brainard), should pay a price in terms of losses in credibility, transparency and control.

However, this argument is only valid if there is an explicit role for expectations in the economic system. We expect therefore, the merits of the two-step rule to manifest themselves only in a forward-looking system. We examine this next.

3.2 A Forward Looking Model

Most of the attempts to examine the effects of uncertainty in a dynamic framework rely on a backward looking set-up (Söderstöm, 2002, Srour, 1999, Craine 1979). The somehow surprising result, from the point of view of inflation targeting proponents, is that uncertainty in the structure of the economy implies that achieving the target is not optimal as doing so may lead to instability in the system. This seems at odds with the general perception that the main advantage of inflation targeting is that it stabilizes expectations. But this is relevant, as already mentioned, only if expectations are an important determinant in the economic system. The importance of this distinction is evident if we apply our rules to a standard New Keynesian, but this time forward-looking model, where expectations do indeed play an active role (Woodford 2002, 2003) contrary to the backward looking model in which expectations are a predetermined variable. Our economy is thus described by a pair of log-linear relations:

$$\pi_t = \beta E_t \pi_{t+1} + \alpha y_t + \varepsilon_t \tag{29}$$

$$y_t = E_t y_{t+1} - \gamma \left(i_t - E_t \pi_{t+1}\right) + \eta_t \tag{30}$$

where (29) is an expectations-augmented "AS" relation in which present inflation is a function of the private sector expectations of inflation one period ahead, and (30) is an intertemporal "IS" relation. The coefficients satisfy, $\alpha, \gamma > 0$, and $0 < \beta < 1$. The CB's instrument is the nominal interest rate. Ideally, what we would need to do to check the value of our two-step procedure is to obtain the optimal interest rate rule under the two alternative assumptions as in sections 2.2 and 3. However, it is not immediately obvious how to solve forward-looking models analytically in the presence of uncertainty. Indeed previous attempts to do so, have resorted to numerical simulations (see, *inter alia,* Wieland, 2000, Giannoni, 2002) and do not provide therefore, analytical solutions.

In the absence of a solution technique, we check instead how the two rules derived in the previous section perform in a forward-looking model, aware of the fact that neither of the two are the optimal rules for the given model. The monetary policy rules derived in the context of a backward-looking model above can however, both be formulated in a general Taylor rule format of the following kind:

$$i_t = \bar{i}_t + \phi_\pi \left(\pi_t - \pi^*\right) + \phi_y \left(y_t\right) \tag{31}$$

We restrict our attention to sets of rules that have ϕ_π, $\phi_y \geq 0$. For given values of ϕ_π, ϕ_y we can therefore identify which of the rules, the Brainard rule or the two-step procedure, is more likely to be stable in the current model.

Substituting the general form (31) now in (29) and (30), the above system can be written in state-space form:

$$E_t \mathbf{z}_{t+1} = \widetilde{\mathbf{A}} * \mathbf{z}_t + \widetilde{\mathbf{B}} * \pi^* + \mathbf{K}\mathbf{u}_{t+1}$$

where, $\mathbf{z}_t' = \begin{bmatrix} \pi_t & y_t \end{bmatrix}$, $\mathbf{u}_t' = \begin{bmatrix} \varepsilon_t & \eta_t & \bar{i}_t \end{bmatrix}$ and

$$\widetilde{\mathbf{A}} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\alpha}{\beta} \\ -\frac{\gamma}{\beta} + \gamma\phi_\pi & \frac{\alpha\gamma}{\beta} + 1 + \gamma\phi_y \end{pmatrix}, \widetilde{\mathbf{B}} = \begin{bmatrix} 0 \\ -\gamma\phi_\pi \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\gamma & -c \end{bmatrix}$$

Proposition 2 *Forward-looking systems in which expectations play a substantial role are served better by rules that aim directly at controlling them.*

Proof 2: The stability of the system requires that the eigenvalues of matrix $\mathbf{A}$ (given by the solutions of the characteristic equation) are outside the unit circle (Blanchard and Kahn, 1980). In other words,

$$\left| \frac{tr(\widetilde{\mathbf{A}}) \pm \sqrt{tr(\widetilde{\mathbf{A}})^2 - 4\det(\widetilde{\mathbf{A}})}}{2} \right| > 1$$

which, given our parameter restrictions and the nature of the problem, reduces to the following three conditions (see Appendix C for a detailed derivation):

$$\det(\breve{\mathbf{A}}) > 0, \qquad \det(\breve{\mathbf{A}}) - tr(\breve{\mathbf{A}}) > -1, \qquad \det(\breve{\mathbf{A}}) + tr(\breve{\mathbf{A}}) > -1$$

This in turns requires:

$$\left(\frac{1-\beta}{\alpha}\right)\phi_y + \phi_\pi > 1 \tag{32}$$

Stability condition (32) provides a general condition for selecting monetary policy rules. The specific monetary policy rules derived in a backward-looking set-up (from 17 and 19) are:

$$\begin{array}{lll} i_{t,BR}: & \phi_{\pi,BR} = \frac{\bar{c}}{a(\bar{c}^2+\sigma^2)}, & \phi_{y,BR} = \frac{\bar{c}}{\bar{c}^2+\sigma^2}(1+b) \\ i_{t,TS}: & \phi_{\pi,TS} = \frac{1}{a\bar{c}}, & \phi_{y,TS} = \frac{(1+b)}{\bar{c}} \end{array}$$

When applying the two-step inflation targeting rule, condition (32) is always respected, since $\frac{1}{a\bar{c}} \geq 1$, independently of the level of uncertainty faced by the CB. On the other hand, using a more cautious rule à la Brainard, which makes the response of the instrument to a deviation from the target decreasing in uncertainty, the system is stable if and only if: .

$$\sigma_c^2 < \frac{1}{ac} + \left(\frac{1-\beta}{\alpha}\right)\left(\frac{1+b}{c}\right) \tag{33}$$

The importance of condition (33) is that it shows the limits of Brainard's analysis when applied in a forward-looking framework. The Brainard result is thus shown to be specific to the backward-looking nature of the model in which it has been analyzed. In a forward-looking model, Brainard's recommendation of being cautious could well lead to losing control of the system, as expectations of the private sector become independent of the policy rule followed by the CB. The main implication of our analysis is the emphasis on identifying the role of expectations. If expectations happen to play a role in the way the economic system operates, then it pays for policy makers to try and tie them down to something which is clear and transparent. This will help them achieve their objectives, quicker and more effectively.

4 Conclusions

We have examined the implications of uncertainty for inflation targeting in a dynamic set-up. Using a standard inflation forecast targeting set-up, we have compared Brainard's conservative principle with a more active monetary policy rule derived from a two-step maximization procedure. The analysis shows that the Brainard attenuation effect is optimal only because of the backward-looking framework in which this principle is usually analyzed. In fact, if expectations do not play any role in determining equilibrium, any rule which tries to control expectations is by definition, inefficient. On the other hand, we show that the Brainard conservative principle could produce instability in a forward-looking model , because of its failure to control expectations. A more aggressive rule, like the one suggested in the paper, can instead increase stability because of its direct targeting of expectations, despite the uncertainty in the transmission parameters.

References

[1] Bernanke, B. and F. Mishkin, 1997. Inflation Targeting: A New Framework for Monetary Policy?, Journal of Economic Perspectives, Vol 11, No 2, 97-116.

[2] Bernanke, B, T. Lauback, F. Mishkin and A. Posen, 1999. Inflation Targeting: Lessons from the International Experience, Princeton University Press.

[3] Blanchard, O. and C. Kahn, 1980. The Solution of Linear Difference Models Under Rational Expectations, Econometrica, Vol. 48, No. 5, July.

[4] Brainard, W., 1967. Uncertainty and the Effectiveness of Policy, American Economic Review, 57, 411-25.

[5] Craine, R., 1979. Optimal Monetary Policy with Uncertainty, Journal of Economic Dynamics and Control, 1, 59-83.

[6] Demertzis, M. and N. Viegi, 2007. Inflation Targeting: A Framework for Communication, DNB Working Paper, No. 142.

[7] Demertzis, M. and N. Viegi, 2008. 'Inflation Targets as Focal Points', International Journal of Central Banking, Vol. 4, No. 1, March, 55-87.

[8] Giannoni, Marc P., 2002. Does Model Uncertainty Justify Caution? Robust Optimal Monetary Policy in a Forward-looking Model, Macroeconomic Dynamics, 6, 111-144.

[9] Gürkaynak, R., A.T. Levin and E.T. Swanson, 2006. Does Inflation Targeting Anchor Long-Run Inflation Expectations? Evidence from Long-Term Bond Yields in the US, UK and Sweden, Federal Reserve Bank of San Francisco, Working Paper 09.

[10] Holly, S. and A. Hughes Hallett, 1989. Optimal Control, Expectations and Uncertainty, Cambridge University Press.

[11] Johnson, D. R., 2002. The Effect of Inflation Targeting on the Behavior of Expected Inflation: Evidence from an 11 Country Panel, Journal of Monetary Economics, 49, 1521-1538.

[12] Levin, A., F.M. Natalucci and J.M. Piger, 2004. The Macroeconomic Effects of Inflation Targeting, Federal Reserve Bank of St. Louis Review, Vol. 86, 4, 51-80.

[13] Mishkin, F S. and K. Schmidt-Hebbel, 2001. One Decade of Inflation Targeting in the World: What Do We Know and What Do We Need to Know? in Norman Loayza and Raimundo Soto, (eds), Inflation Targeting: Design, Performance, Challenges, Central Bank of Chile, Santiago, 117-219.

[14] Mishkin, F S. and K. Schmidt-Hebbel, 2007. Does Inflation Targeting Make a Difference?, NBER Working Paper, No. 12876, January.

[15] Rudd, J. and K. Whelan, 2006. Can Rational Expectations Sticky-Price Models Explain Inflation Dynamics?, American Economic Review, Volume 96, 303-320.

[16] Södeström, U., 2002. Monetary Policy with Uncertain Parameters, Scandinavian Journal of Economics, 104, 125-145.

[17] Srour, G., 1999. Inflation Targeting under Uncertainty Technical Report 85, Bank of Canada, April.

[18] Svensson, L., 1999a. Inflation Targeting: some extensions Scandinavian Journal of Economics, 101, 337-361.

[19] Svensson, L., 1999b. Inflation Targeting as a Monetary Policy Rule, Journal of Monetary Economics, 43, 607-654.

[20] Wieland, V., 2000. Monetary Policy, Parameter Uncertainty and Optimal Learning, Journal of Monetary Economics, August.

[21] Woodford, M., 2002. The Taylor Rule and Optimal Monetary Policy, American Economic Review, 91, No.2, 232-237.

[22] Woodford, M., 2003, *Interest and Prices*, Princeton University Press.

APPENDICES

A Two-Step Inflation Targeting

The CB can now improve on the previous result by using its information advantage, namely the knowledge of the shock that has hit the economy[3]. We put forward an example of a rule that aim directly to align expectations to the target. We do this in a two-step optimization procedure according to which the Central Bank, first identifies the optimal rule as a function of $\pi^* + \theta$, where θ is a deviation from the target decided after the shock is observed, and second, it optimizes with respect to this deviation, aiming to close the gap from its objectives which arise due to the existence of uncertainty[4]. The following two sections describe the two-step procedure in greater detail.

A.1 Step 1

In the first step, and after the shock has occurred, the monetary policy authorities identify the optimal policy rule as a function of an auxiliary target $(\pi^* + \theta)$. Formally this means optimizing the following objective function

$$\min_{i_t} E(L) = \delta^2 \frac{1}{2} \left\{ \left[\bar{\pi}_{t+2} - (\pi^* + \theta) \right]^2 + \sigma^2_{\pi_{t+2}} \right\} \tag{A1}$$

subject to the system of equations (1) and (2). In other words it optimizes the expected value of the following auxiliary objective function[5]:

$$\begin{aligned} \min_{i_t} E(L) \quad = \quad & \delta^2 \frac{1}{2} \Big\{ \left[\pi_t + aby_t - a\bar{c}i_t - (\pi^* + \theta) \right]^2 \\ & + \sigma_c^2 a^2 (i_t - \pi_{t+1|t}) + 2\sigma_\varepsilon^2 + a^2 \sigma_\eta^2 \Big\} \end{aligned} \tag{A2}$$

Optimizing (A1) produces an optimal rule as a function of θ. The monetary policy reaction function and the resulting inflation forecast for period $t+2$ are:

$$i_t = \pi_{t+1|t} + \frac{\bar{c}}{a\left(\bar{c}^2 + \sigma_c^2\right)} \left[\pi_t - (\pi^* + \theta) \right] + \frac{\bar{c}}{\left(\bar{c}^2 + \sigma_c^2\right)} (1+b) y_t \tag{A3}$$

and

$$\pi^e_{t+2} = \frac{\sigma_c^2}{\bar{c}^2 + \sigma_c^2} \pi_t + \frac{\bar{c}^2}{\bar{c}^2 + \sigma_c^2} (\pi^* + \theta) + \frac{a(1+b)\sigma_c^2}{\bar{c}^2 + \sigma_c^2} y_t \tag{A4}$$

[3] As already mentioned, this presumes that the private sector forms expectations first, a shock occurs next and the CB reacts by choosing that interest rate which optimises the conditional expectation of its loss function.

[4] In effect this amounts to giving the Central Bank an extra instrument while the number of targets remains the same.

[5] The expected value of the objective function is conditional on the shocks, omitted here for simplicity.

The above two equations imply that for a given level of uncertainty, the CB will choose to deviate, at first instance, from its ultimate target π^* by a parameter θ.

A.2 Step 2

The degree of deviation θ is chosen optimally. In other words, the CB chooses θ in full knowledge of the extent of uncertainty and the size of the shock, and aims to maximize the probability of achieving its true objectives. In other words, since inflation expectations move away from the target as uncertainty increases, the deviation term θ will move to close that gap. Similarly, the instrument will In that respect θ is therefore, an auxiliary step, necessary in order to make full use of the information available to the bank. The derived rules from Step 1 (A3) and (A4) are now substituted into the objective function of the Central Bank:

$$\min_{\theta} E(L) = \delta^2 E_t \left[\frac{1}{2} (\pi_{t+2} - \pi^*)^2 \right] \tag{A9}$$

to produce

$$\min_{\theta} E(L) = f(\theta, \sigma_c^2, y_t, \pi_t) \tag{A10}$$

Given the rules, the aim of the CB is to find the optimal value for θ, contingent on the economy's past history and the perceived uncertainty of the transmission of policies, i.e.:

$$\theta(\sigma_c^2, y_t, \pi_t) = \arg\min_{\theta} L$$

which in its analytical form is

$$\theta = -\frac{\sigma_c^2}{\overline{c}^2} [\pi_t - \pi^* + a(1+b) y_t] \tag{A11}$$

As uncertainty decreases, the deviations from π^* decrease as well, such that at the limit they become zero, i.e.

$$\lim_{\sigma_a^2 \to 0} (\theta) = 0$$

Substituting the analytical solutions for θ into (A3) - (A4) produces the following interest rate rule and inflation forecast:

$$i_{t,TS} = \pi_{t+1|t} + \frac{1}{a\overline{c}} (\pi_t - \pi^*) + \frac{(1+b)}{\overline{c}} y_t \tag{A12}$$

$$\pi^e_{t+2|t,TS} = \pi^* \tag{A13}$$

The rules achieved are similar to those attained with no uncertainty (with c replaced by $\overline{c}$). This demonstrates that by varying the target optimally, uncertainty in the transmission process is neutralized. This however, is an *ex ante* result. As we will show next, this happens at the expense of using i_t more actively, thereby introducing greater variability in the system. It follows that the obvious *ex ante* benefits of the two-step procedure do not necessarily carry also *ex post*. To evaluate these, we turn to an analysis of the stability properties of the system.

B Analyzing the stability of the systems

The distinction between *ex ante* and *ex post* that we have emphasized in the analysis undertaken implies that although the two-step procedure does actually deliver the forecast target intended, it is not necessarily the case that the system does better overall, *ex post*. Our simulations show this very clearly for a backward-looking model. In fact what we see is that the two-step scenario generates instability in the system. This is what we analyze next by looking at the stability properties of the system. For completeness sake, we examine the properties for all three cases shown in the main text.

B.1 The System with no Parameter Uncertainty

A system with no parameter uncertainty is stable due to full controllability of the system. Substituting (10) in (2) we have:

$$y_{t+1} = b\left(y_t\right) - c\left(\frac{1}{ac}\left(\pi_t - \pi^*\right) + \frac{1+b}{c} y_t\right) + \eta_{t+1} \tag{B1}$$

The system of equations (1) and (B1) can then be summarized as:

$$\begin{aligned} \pi_{t+1} &= \pi_t + a\left(y_t\right) + \varepsilon_{t+1} && \text{(B2)} \\ y_{t+1} &= -\frac{1}{a}\pi_t - \left(y_t\right) + \frac{1}{a}\pi^* + \eta_{t+1} && \text{(B3)} \end{aligned}$$

or in a matrix form, this can be written as

$$\begin{pmatrix} \pi_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & a \\ -\frac{1}{a} & -1 \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{a} \end{pmatrix} \pi^* + \begin{pmatrix} \varepsilon_{t+1} \\ \eta_{t+1} \end{pmatrix}$$

which has the following state-space representation form, $\mathbf{z}_{t+1} = \mathbf{A}_{CE} * \mathbf{z}_t + \mathbf{B}_{CE} * \pi^* + \mathbf{u}_{t+1}$. Analyzing the stability of the system implies looking at the eigenvalues of matrix $\mathbf{A}_{CE}$. The determinant of the characteristic matrix $|\mathbf{D}_{CE}|$ is

$\mathbf{D}_{CE} = \begin{vmatrix} 1-\lambda & a \\ -\frac{1}{a} & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - a\left(-\frac{1}{a}\right) = -(1-\lambda)(1+\lambda) + 1 = -\left(1-\lambda^2\right) + 1 = \lambda^2 = 0$

and the two eigenvalues are therefore identical and equal to zero.

B.2 The system with Brainard Uncertainty

Introducing now multiplicative uncertainty in parameter c implies that the optimal rule is (17) which we can substitute in (2) to get:

$$y_{t+1} = b(y_t) - c_i \left(\frac{\bar{c}}{a(\bar{c}^2 + \sigma_c^2)} (\pi_t - \pi^*) + \frac{\bar{c}}{\bar{c}^2 + \sigma_c^2} (1+b) y_t \right) + \eta_{t+1} \quad \text{(B4)}$$

where $\bar{c}$ is the average transmission effect of the instrument expected and c_i is the random effect drawn each time. We re-specify c_i as a deviation from its mean such that $c_i = \bar{c} + \varphi_i$. For simplicity we also call $k = \bar{c}^2 + \sigma_c^2$. The system of equations (6) and (2) can be re-written as

$$\begin{aligned} \pi_{t+1} &= \pi_t + a(y_t) + \varepsilon_{t+1} && \text{(B5)} \\ y_{t+1} &= \frac{-(\bar{c}+\varphi_i)\bar{c}}{ak}\pi_t + \left[b - \frac{(\bar{c}+\varphi_i)\bar{c}}{a}(1+b) \right](y_t) + \frac{(\bar{c}+\varphi_i)\bar{c}}{ak}\pi^* + \eta_{t+1} && \text{or} \\ y_{t+1} &= \frac{-(\bar{c}+\varphi_i)\bar{c}}{ak}\pi_t + \left[\frac{kb - (\bar{c}+\varphi_i)\bar{c}}{k}(1+b) \right](y_t) + \frac{(\bar{c}+\varphi_i)\bar{c}}{ak}\pi^* + \eta_{t+1} && \text{(B6)} \end{aligned}$$

In a matrix form this can be written as

$$\begin{aligned} \begin{pmatrix} \pi_{t+1} \\ y_{t+1} \end{pmatrix} &= \begin{pmatrix} 1 & a \\ \frac{-(\bar{c}+\varphi_i)\bar{c}}{a(\bar{c}^2+\sigma_c^2)} & b - \frac{(\bar{c}+\varphi_i)\bar{c}}{(\bar{c}^2+\sigma_c^2)}(1+b) \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ \frac{(\bar{c}+\varphi_i)\bar{c}}{a(\bar{c}^2+\sigma_c^2)} \end{pmatrix} \pi^* + \begin{pmatrix} \varepsilon_{t+1} \\ \eta_{t+1} \end{pmatrix} \end{aligned}$$

which has the following state-space representation form: $\mathbf{z}_{t+1} = \mathbf{A}_{BR} * \mathbf{z}_t + \mathbf{B}_{BR} * \pi^* + \mathbf{u}_{t+1}$. Again, to analyze the stability of the system implies looking at the eigenvalues of matrix $\mathbf{A}_{BR}$.

B.3 Two-Step Inflation Targeting

Analyzing now the stability of the system derived with the two-step inflation targeting procedure, implies that we can substitute the Certainty rule, (10) in (2) but this time replacing parameter c with $\bar{c}$.

$$y_{t+1} = b(y_t) - c_i \left(\frac{1}{a\bar{c}} (\pi_t - \pi^*) + \frac{1+b}{\bar{c}} y_t \right) + \eta_{t+1} \quad \text{(B7)}$$

Similarly to above we specify c_i as a deviation from its mean such that $c_i = \bar{c} + \varphi_i$. For a given draw of parameter c_i the system is thus:

$$\begin{aligned} \pi_{t+1} &= \pi_t + a(y_t) + \varepsilon_{t+1} \\ y_{t+1} &= b(y_t) - (\bar{c}+\varphi_i)\left(\frac{1}{a\bar{c}}(\pi_t - \pi^*) + \frac{1+b}{\bar{c}} y_t \right) + \eta_{t+1} \end{aligned}$$

or

$$\pi_{t+1} = \pi_t + a(y_t) + \varepsilon_{t+1} \tag{B8}$$

$$y_{t+1} = -\left[\frac{\bar{c}+\varphi_i}{a\bar{c}}\right]\pi_t - \left[\frac{\bar{c}+\varphi_i(1+b)}{\bar{c}}\right](y_t) + \left(\frac{\bar{c}+\varphi_i}{a\bar{c}}\right)\pi^* + \eta_{t+1} \tag{B9}$$

Or in matrix form:

$$\begin{pmatrix} \pi_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & a \\ -\frac{\bar{c}+\varphi_i}{a\bar{c}} & -\frac{\bar{c}+\varphi_i(1+b)}{\bar{c}} \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\bar{c}+\varphi_i}{a\bar{c}} \end{pmatrix} \pi^* + \begin{pmatrix} \varepsilon_{t+1} \\ \eta_{t+1} \end{pmatrix}$$

and in simplified notation: $\mathbf{z}_{t+1} = \mathbf{A}_{TS} * \mathbf{z}_t + \mathbf{B}_{TS} * \pi^* + \mathbf{u}_{t+1}$.

C A Forward Looking Model

We now consider a forward-looking model:

$$\begin{aligned} \pi_t &= \beta E_t \pi_{t+1} + \alpha y_t + \varepsilon_t \\ y_t &= E_t y_{t+1} - \gamma(i_t - E_t \pi_{t+1}) + \eta_t \end{aligned}$$

The CB's instrument is the nominal interest rate but the private sector forms expectations at time t for inflation at time $t+1$. This can be re-written as

$$\begin{aligned} E_t \pi_{t+1} &= \frac{1}{\beta}(\pi_t - \alpha y_t - \varepsilon_t) \\ \gamma E_t \pi_{t+1} + E_t y_{t+1} &= y_t + \gamma i_t - \eta_t \end{aligned}$$

where parameters α, β, γ are all positive. We can then examine the stability of the system for any interest rate rule, i. The monetary policy rules derived in the context of a backward-looking model above can both be formulated in a general Taylor rule format of the kind:

$$i = \phi_\pi(\pi_t - \pi^*) + \phi_y(y_t) \tag{C1}$$

We restrict our attention to sets of rules that have ϕ_π, $\phi_y \geq 0$. For given values of ϕ_π, ϕ_y we can therefore identify which of the rules, that identified by Brainard or the two-step target, is more likely to be stable.
Substituting the general rule (C1) now in the above system can be written in matrix form:

$$\begin{aligned} \gamma E_t \pi_{t+1} + E_t y_{t+1} &= y_t + \gamma\left[\phi_\pi(\pi_t - \pi^*) + \phi_y(y_t)\right] - \eta_t \quad \Leftrightarrow \\ \gamma E_t \pi_{t+1} + E_t y_{t+1} &= \gamma\phi_\pi \pi_t + \left(1 + \gamma\phi_y\right) y_t - \gamma\phi_\pi \pi^* - \eta_t \end{aligned}$$

And in matrix terms:

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} E_t\pi_{t+1} \\ E_t y_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\alpha}{\beta} \\ \gamma\phi_\pi & 1+\gamma\phi_y \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} + \begin{pmatrix} 0 \\ -\gamma\phi_\pi \end{pmatrix} \pi^* + \begin{pmatrix} -\frac{1}{\beta} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \varepsilon_{t+1} \\ \eta_{t+1} \end{pmatrix}$$

or in simplified form

$$\begin{aligned} \mathbf{C}E_t\mathbf{z}_{t+1} &= \mathbf{A}*\mathbf{z}_t + \mathbf{B}*\pi^* + \mathbf{K}\mathbf{u}_{t+1} \qquad \text{and} \\ E_t\mathbf{z}_{t+1} &= \breve{\mathbf{A}}*\mathbf{z}_t + \breve{\mathbf{B}}*\pi^* + \breve{\mathbf{K}}\mathbf{u}_{t+1} \end{aligned}$$

where $\breve{\mathbf{A}} = \mathbf{C}^{-1}\mathbf{A}$, $\breve{\mathbf{B}} = \mathbf{C}^{-1}\mathbf{B}$ and $\breve{\mathbf{K}} = \mathbf{C}^{-1}\mathbf{K}$. The stability of the system requires that the eigenvalues of matrix $\breve{\mathbf{A}} = \mathbf{C}^{-1}\mathbf{A}$ be outside the unit circle. Matrix $\breve{\mathbf{A}}$ is

$$\breve{\mathbf{A}} \equiv \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\beta} & -\frac{\alpha}{\beta} \\ \gamma\phi_\pi & 1+\gamma\phi_y \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\alpha}{\beta} \\ -\frac{\gamma}{\beta}+\gamma\phi_\pi & \frac{\alpha\gamma}{\beta}+1+\gamma\phi_y \end{pmatrix}$$

where

$$\mathbf{C}^{-1} = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}$$

The eigenvalues are given by the solution of the characteristic equation and are required to have both values outside the unit circle for stability. In other words,

$$\left| \frac{tr(\breve{\mathbf{A}}) \pm \sqrt{tr(\breve{\mathbf{A}})^2 - 4\det(\breve{\mathbf{A}})}}{2} \right| > 1$$

which is equivalent to two conditions:

$$\frac{tr(\breve{\mathbf{A}}) \pm \sqrt{tr(\breve{\mathbf{A}})^2 - 4\det(\breve{\mathbf{A}})}}{2} > 1 \tag{C2}$$

$$\frac{tr(\breve{\mathbf{A}}) \pm \sqrt{tr(\breve{\mathbf{A}})^2 - 4\det(\breve{\mathbf{A}})}}{2} < -1 \tag{C3}$$

Conditions (C2) and (C3) are mutually exclusive in pairs.

C.1 Solving for (C2)

$$\begin{aligned} \frac{tr(\breve{\mathbf{A}}) \pm \sqrt{tr(\breve{\mathbf{A}})^2 - 4\det(\breve{\mathbf{A}})}}{2} &> 1 \\ \frac{tr(\breve{\mathbf{A}})}{2} \pm \sqrt{\frac{tr(\breve{\mathbf{A}})^2}{4} - \det(\breve{\mathbf{A}})} &> 1 \end{aligned}$$

We analyze the two solutions separately:

$$\begin{aligned} \frac{tr(\breve{\mathbf{A}})}{2} + \sqrt{\frac{tr(\breve{\mathbf{A}})^2}{4} - \det(\breve{\mathbf{A}})} &> 1 \\ \det(\breve{\mathbf{A}}) - tr(\breve{\mathbf{A}}) &< -1 \end{aligned}$$

and

$$\begin{aligned} \frac{tr(\breve{\mathbf{A}})}{2} - \sqrt{\frac{tr(\breve{\mathbf{A}})^2}{4} - \det(\breve{\mathbf{A}})} &> 1 \\ -1 &< \det(\breve{\mathbf{A}}) - tr(\breve{\mathbf{A}}) \end{aligned}$$

C.2 Solving for (C3)

$$\begin{aligned} \frac{tr(\breve{\mathbf{A}}) \pm \sqrt{tr(\breve{\mathbf{A}})^2 - 4\det(\breve{\mathbf{A}})}}{2} &< -1 \\ \frac{tr(\breve{\mathbf{A}})}{2} \pm \sqrt{\frac{tr(\breve{\mathbf{A}})^2}{4} - \det(\breve{\mathbf{A}})} &< -1 \end{aligned}$$

Again we analyze the two solutions separately:

$$\begin{aligned} \frac{tr(\breve{\mathbf{A}})}{2} + \sqrt{\frac{tr(\breve{\mathbf{A}})^2}{4} - \det(\breve{\mathbf{A}})} &< -1 \\ -1 &< tr(\breve{\mathbf{A}}) + \det(\breve{\mathbf{A}}) \end{aligned}$$

and the minus

$$\begin{aligned} \frac{tr(\breve{\mathbf{A}})}{2} - \sqrt{\frac{tr(\breve{\mathbf{A}})^2}{4} - \det(\breve{\mathbf{A}})} &< -1 \\ -1 &> tr(\breve{\mathbf{A}}) + \det(\breve{\mathbf{A}}) \end{aligned}$$

C.3 Analyzing the stability

The above analysis produces therefore four conditions for stability:

$$\begin{aligned} -1 &> \det(\breve{\mathbf{A}}) - tr(\breve{\mathbf{A}}) && \text{(C4)} \\ -1 &< \det(\breve{\mathbf{A}}) - tr(\breve{\mathbf{A}}) && \text{(C5)} \\ -1 &< \det(\breve{\mathbf{A}}) + tr(\breve{\mathbf{A}}) && \text{(C6)} \\ -1 &> \det(\breve{\mathbf{A}}) + tr(\breve{\mathbf{A}}) && \text{(C7)} \end{aligned}$$

We calculate next the $\det(\breve{\mathbf{A}})$ and $tr(\breve{\mathbf{A}})$.

$$\begin{aligned} \det(\breve{\mathbf{A}}) &= \frac{1}{\beta}\left(\frac{\alpha\gamma}{\beta} + 1 + \gamma\phi_y\right) - \left(-\frac{\gamma}{\beta} + \gamma\phi_\pi\right)\left(-\frac{\alpha}{\beta}\right) \\ &= \frac{\alpha\gamma}{\beta^2} + \frac{1}{\beta} + \frac{\gamma\phi_y}{\beta} - \frac{\alpha\gamma}{\beta^2} + \frac{\alpha\gamma\phi_\pi}{\beta} \\ &= \frac{1}{\beta} + \frac{\gamma\phi_y}{\beta} + \frac{\alpha\gamma\phi_\pi}{\beta} \\ &= \frac{1}{\beta}\left(1 + \gamma\phi_y + \alpha\gamma\phi_\pi\right) > 0 \end{aligned}$$

and the trace

$$tr(\breve{\mathbf{A}}) = \frac{1}{\beta} + \frac{\alpha\gamma}{\beta} + 1 + \gamma\phi_y > 0$$

The parameter restrictions imply that condition (C6) is **always** true. Condition (C7) is therefore, **never** true. Furthermore, since (C6) is always true and in its original form shows that $\frac{tr(\breve{\mathbf{A}})}{2} + \sqrt{\frac{tr(\breve{\mathbf{A}})^2}{4} - \det(\breve{\mathbf{A}})} < -1$, then it follows that the expression $\frac{tr(\breve{\mathbf{A}})}{2} + \sqrt{\frac{tr(\breve{\mathbf{A}})^2}{4} - \det(\breve{\mathbf{A}})} > 1$ and therefore (C4) is also never true. By implication condition (C5) holds. We examine then the parameter restrictions required for this to be true:

$$\begin{aligned} \det(\breve{\mathbf{A}}) - tr(\breve{\mathbf{A}}) &> -1 \\ \frac{1}{\beta}\left(1 + \gamma\phi_y + \alpha\gamma\phi_\pi\right) - \frac{1}{\beta} - \frac{\alpha\gamma}{\beta} - 1 - \gamma\phi_y &> -1 \\ \left(\frac{1-\beta}{\alpha}\right)\phi_y + \phi_\pi &> 1 \end{aligned}$$

In summary the conditions are[6]

$$\det(\breve{\mathbf{A}}) > 0, \quad \det(\breve{\mathbf{A}}) - tr(\breve{\mathbf{A}}) > -1, \quad \det(\breve{\mathbf{A}}) + tr(\breve{\mathbf{A}}) > -1 \qquad \text{(C8)}$$

[6] See Woodford, 2002.

Our parameter restrictions implies that the second is always true, whereas the first holds when

$$\left(\frac{1-\beta}{\alpha}\right)\phi_y+\phi_\pi>1 \tag{C9}$$

We can therefore compare the likelihood with which the rules $i_{t,BR}$ and $i_{t,TS}$ are unstable. Condition (C9) therefore, is likely to be true for higher values of ϕ_y, ϕ_π which is the case for $i_{t,TS}$.

$$\begin{aligned}
i_{t,BR} &: \qquad \phi_{y,BR}=\frac{-\bar{c}ab}{\bar{c}^2+\sigma^2},\quad \phi_{\pi,BR}=\frac{\bar{c}\,(a+b)}{a\,(\bar{c}+\sigma^2)}\\
i_{t,TS} &: \qquad \phi_{y,TS}=\frac{a\bar{c}+1}{a\bar{c}},\quad \phi_{\pi,TS}=\frac{b}{\bar{c}}
\end{aligned}$$

and the ratio between them is

$$\begin{aligned}
\frac{\phi_{y,BR}}{\phi_{\pi,BR}} &= \frac{\frac{-\bar{c}ab}{\bar{c}^2+\sigma^2}}{\frac{\bar{c}(a+b)}{a(\bar{c}+\sigma^2)}}=\frac{-\bar{c}a^2b\left(\bar{c}+\sigma^2\right)}{\left(\bar{c}^2+\sigma^2\right)\bar{c}\,(a+b)}\\
\frac{\phi_{y,TS}}{\phi_{\pi,TS}} &= \frac{\frac{b}{\bar{c}}}{\frac{a\bar{c}+1}{a\bar{c}}}=\frac{ab}{a\bar{c}+1}
\end{aligned}$$